

Local Null Controllability of a Chemotaxis System of Parabolic-Elliptic Type*

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Abstract

In this paper, we are concerned with the controllability of a chemotaxis system of parabolic-elliptic type. By linearizing the nonlinear system into two separated linear equations to bypass the obstacle caused by the nonlinear drift term, we establish the local null controllability of the original nonlinear system. The approach is different from the usual way of treating the coupled parabolic systems.

Keywords: Local null controllability, chemotaxis system, parabolic-elliptic type, Kakutani's fixed point theorem.

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1 Introduction and main result

In this paper, we are concerned with a controlled initial-boundary value system of parabolic-elliptic type

$$\begin{cases} \partial_t u = \nabla \cdot (\nabla u - \chi u \nabla v) + \mathbf{1}_\omega f & \text{in } \Omega \times (0, T), \\ 0 = \Delta v - \gamma v + \delta u & \text{in } \Omega \times (0, T), \\ \partial_\nu u = 0, \partial_\nu v = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & x \in \Omega, \end{cases} \quad (1.1)$$

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where and henceforth u and v are shorthands for states $u(x, t)$ and $v(x, t)$ at time spacial position $x \in \Omega$ and time $t \geq 0$, $\partial_t = \partial/\partial t$, $\partial_\nu = \partial/\partial \nu$ is the directional derivative along the outward unit normal vector ν on $\partial\Omega$, $\mathbf{1}_\omega$ stands for the characteristic function of ω , f is the control function, u_0 is the initial value, χ, γ and δ are given positive constants.

The system (1.1) without the control (i.e., $f \equiv 0$) is a simple chemotaxis system, which was addressed by Keller and Segel [14] as a model to describe the aggregation process in slime mold morphogenesis, assuming that the cells emit directly the chemoattractant which is immediately diffused. The unknown function u then denotes the cell density, whereas v represents the concentration of the chemoattractant. The validity of Keller-Segel's chemotaxis system is supported by some experiments on the *Escherichia Coli* bacteria and other interesting physical interpretations. In fact, it has been extensively involved in many medical and biological applications as well as some relevant areas such as ecology and environment sciences. Moreover, because the model has a rich structure from mathematical point of view, it deserves to be challenged. Actually, some special but interesting cases have been studied such as the aggregation, the blow-up of solutions, and the chemotactic collapse. Some significant results have been achieved from different perspectives. We refer to [11] (also [12]) for a survey where a quite complete bibliography on the topic is included.

In this paper, we study the Keller-Segel system from the controllability point of view. We say that the system (1.1) is *locally null controllable at time T* , if there exists a neighborhood of the origin such that for any initial data u_0 belonging to this neighborhood, the solution (u, v) of (1.1) produced by corresponding control function f satisfies $u(x, T) = 0$ for almost all $x \in \Omega$, where the neighborhood and the control function space will be specified later. Here, we consider the local null controllability instead of the exact null controllability. The reason being interested is that the solutions of Keller-Segel system may blow up in either finite time or infinite time, which is shown in [10] for the 3-d case. The 2-d case is even more interesting and attractive. Actually, it has been found for the 2-d case that the solution exists globally in finite time when the mass of the initial value is less than a threshold value, while the solution will blow up either in finite or in infinite time when the mass of initial value is larger than the threshold value (see, e.g., [13]).

The study of the controllability for parabolic equations has been thriving in the past decade, see for instance [5, 8] and the references therein. Among them, a special interest is on the controllability of coupled parabolic systems. For coupled systems, the most practical situation is to impose the control force on one equation, which has attracted intensive attention in the last few years. We refer to a survey paper [2] wherein abundant references are provided.

However, to the best of our knowledge, very few results are available to the control problems of the Keller-Segel system (1.1) where a parabolic equation is coupled with an elliptic equation through a drift term. Very recently, the null controllability of a kind of nonlinear parabolic-elliptic system of the following is considered in [7]:

$$\begin{cases} \partial_t y - \Delta y = F(y, z) + \mathbf{1}_\omega f & \text{in } \Omega \times (0, T), \\ -\Delta z = f(y, z) & \text{in } \Omega \times (0, T), \end{cases}$$

where $F(y, z)$ and $f(y, z)$ are nonlinear terms. In [16], an optimal control problem of the system (1.1) with the control to be imposed on the second equation is considered. Our previous work [9] is the first work that considers the local exact controllability of a type of Keller-Segel system where a parabolic equation couples with another parabolic equation. In the present work, we attempt the controllability of system (1.1), which is probably the first work for this system. In the system (1.1), since the drift term $-\chi \nabla \cdot (u \nabla v)$ destroys some good properties of the diffusion operator which ensure the regularity of the system, much more mathematical difficulties than the aforementioned coupled parabolic systems are caused. These include the regularity, the estimation of the “observability inequality”, and many others. The usual way to establish the controllability of a nonlinear system is to linearize the nonlinear system into some coupled linear ones. Then, by the combination of the controllability result of the linearized system and some fixed point results, one is able to establish the controllability of the nonlinear system. We refer to the typical works [7, 9] for the approach of this kind. In this paper, we investigate, however, the controllability of system (1.1) in a different way motivated intuitively by the special mathematical structure of system (1.1). We may decompose this nonlinear system into two separated linear systems: one is a controlled parabolic system, and another one is an irrelevant elliptic equation. In such a way, we can bypass the obstacle caused by the nonlinear drift term. This technique would be useful for other coupled systems like drift-diffusion equations from the semiconductor device.

Throughout the paper, $\Omega \subset \mathbb{R}^N (N \geq 1)$ stands for a bounded domain with smooth boundary $\partial\Omega$, ω is a nonempty open subset of Ω , and $T > 0$. $Q = \Omega \times (0, T)$, $\Sigma = \partial\Omega \times (0, T)$, and $Q_\omega = \omega \times (0, T)$. The norms of the usual Lebesgue function spaces $L^p(\Omega)$ and $L^p(Q)$ are denoted by $|\cdot|_p$ and $\|\cdot\|_p$, respectively. $W^{s,q}(\Omega)$, $W_q^{2,1}(Q)$, $C^{2,1}(\bar{Q})$ and $C^\alpha(\bar{\Omega})$ ($s, \alpha \geq 0$, $1 \leq q \leq \infty$) represent the usual Sobolev spaces (see, e.g., [15]). When $q = 2$, $H^m(\Omega) = W^{m,2}(\Omega)$, $m \in \mathbb{N}$. In addition,

$$W(0, T) = \{y | y \in L^2(0, T; H^1(\Omega)), \partial_t y \in L^2(0, T; H^1(\Omega)^*)\}$$

is equipped with the graph norm $\|y\|_{W(0,T)} = \|y\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t y\|_{L^2(0,T;H^1(\Omega)^*)}$, where $H^1(\Omega)^*$ denotes the dual space of $H^1(\Omega)$ and their duality product is denoted by $\langle \cdot, \cdot \rangle$. We also use C to denote a positive constant independent of T , which have different values in different contexts.

Definition 1. A pair of functions (u, v) , with $u \in W(0, T) \cap L^\infty(Q)$ and $v \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q)$, is said to be a *weak solution* of (1.1) if for every $\varphi \in L^2(0, T; H^1(\Omega))$, the following identities hold true:

$$\begin{aligned} \int_0^T \langle \partial_t u, \varphi \rangle dt + \iint_Q [(\nabla u - \chi u \nabla v) \cdot \nabla \varphi - \mathbf{1}_\omega f \varphi] dx dt &= 0, \\ \iint_Q [\nabla v \cdot \nabla \varphi + (\gamma v - \delta u) \varphi] dx dt &= 0. \end{aligned}$$

Now, we are in a position to state the main result of this paper.

Theorem 1.1. *Let $T > 0$. For any initial value u_0 satisfying*

$$|u_0|_\infty \leq e^{-c_1(1+T+\frac{1}{T})}, \quad (1.2)$$

where $c_1 > 0$ is a constant independent of T , there exists a control $f \in L^\infty(Q_\omega)$ such that system (1.1) admits a solution (u, v) satisfying

$$u \in C([0, T]; L^2(\Omega)) \cap L^\infty(Q), v \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q),$$

and $u(x, T) = 0$ for almost all $x \in \Omega$.

Remark 1.1. Theorem 1.1 gives an explicit representation of the initial data with respect to time T . It particularly shows that the shorter of the terminal time T is, the smaller for the initial value for the null controllability of system (1.1). In addition, under the assumption of Theorem 1.1, $\limsup_{t \rightarrow T^-} \|v(\cdot, t)\|_2 = 0$.

We proceed as follows. In section 2, we give some preliminary results. Section 3 is devoted to the null controllability of a scalar parabolic equation, for which the L^∞ -control is obtained and its estimates with respect to time T are also established. The proof of Theorem 1.1 is presented in section 4.

2 Some results for linear equations

In the sequel of the paper, we need some regularity results for linear equations for both parabolic and elliptic types. We first consider the well-posedness of the linear elliptic equation followed by

$$\begin{cases} 0 = \Delta v - \gamma v + \delta \eta & \text{in } \Omega, \\ \partial_\nu v = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

where γ and δ are positive constants. The result of Proposition 2.1 is brought from [1].

Proposition 2.1. *For any $\eta \in L^p(\Omega)$, $p > 1$, Eq. (2.1) admits a unique solution $v \in W^{2,p}(\Omega)$ with*

$$\|v\|_{W^{2,p}(\Omega)} \leq C \|\eta\|_p.$$

Next, we consider the parabolic equation

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (Bu) + F & \text{in } Q, \\ \partial_\nu u = 0 & \text{on } \Sigma, \\ u(x, 0) = u_0(x) & x \in \Omega. \end{cases} \quad (2.2)$$

Proposition 2.2. *Let $B \in L^\infty(Q)^N$ with $B \cdot \nu = 0$ on Σ , $F \in L^\infty(Q)$, and $u_0 \in L^\infty(\Omega)$. Then Eq. (2.2) admits a weak solution $u \in L^\infty(Q)$ with*

$$\|u\|_\infty \leq e^{C\varrho_0} (\|u_0\|_\infty + \|F\|_\infty), \quad (2.3)$$

where $C = C(\Omega)$ is a positive constant depending on Ω , and ϱ_0 is given by

$$\varrho_0 = (1 + \|B\|_\infty^2)(1 + T). \quad (2.4)$$

A similar inequality (2.3) could be found in [15], but here we improve the estimate so that it depends on time T explicitly. To this end, we need the following lemma (see [15, Lemma 5.6, p. 95]).

Lemma 2.1. *Suppose that a sequence $Y_s, s = 0, 1, 2, \dots$ of nonnegative numbers satisfy a recursion relation*

$$Y_{s+1} \leq cb^s Y_s^{1+\varepsilon}, \quad s = 0, 1, 2, \dots$$

with some positive constants c, ε and $b \geq 1$. Then $Y_s \rightarrow 0$ as $s \rightarrow \infty$ provided that

$$Y_0 \leq c^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}}.$$

Proof of Proposition 2.2. Let $(u - k)_+ = \max\{u - k, 0\}$ and $A_k(t) = \text{meas}\{x \in \Omega | u(x, t) > k\}$ for $t \in [0, T]$, where

$$k \geq K_0 = \|F\|_\infty + |u_0|_\infty. \quad (2.5)$$

Multiplying by $(u - k)_+$ the both sides of (2.2), we get, by integration by parts and the Hölder inequality, that

$$\begin{aligned} & \frac{d}{dt} \int_\Omega |(u - k)_+|^2 dx + \int_\Omega |\nabla(u - k)_+|^2 dx \\ & \leq \|B\|_\infty^2 \int_{A_k(t)} u^2 dx + \int_\Omega |(u - k)_+|^2 dx + \int_{A_k(t)} F^2 dx. \\ & \leq (2\|B\|_\infty^2 + 1) \left(\int_\Omega |(u - k)_+|^2 dx + \int_{A_k(t)} k^2 dx \right). \end{aligned}$$

From Gronwall's inequality, it follows that

$$\int_\Omega |(u - k)_+|^2 dx + \int_0^t \int_\Omega |\nabla(u - k)_+|^2 dx dt \leq e^{C\varrho_0} \int_0^T \int_{A_k(t)} k^2 dx dt \quad (2.6)$$

for all $t \in [0, T]$, where and in what follows ϱ_0 is given by (2.4). On the other hand, by Proposition I.3.2 of [6],

$$\|v\|_{\frac{2(N+2)}{N}} \leq C(\Omega)(1 + T)^{\frac{N}{2(N+2)}} \|v\|_{V_2(Q)} \quad (2.7)$$

for any $v \in V_2(Q)$. Here $V_2(Q) = L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ is endowed with its graph norm. Then, (2.6) together with (2.7) gives

$$\|(u - k)_+\|_{\frac{2(N+2)}{N}} \leq e^{C\varrho_0} \int_0^T \int_{A_k(t)} k^2 dx dt. \quad (2.8)$$

Let $\varphi(k) = \text{meas}\{(x, t) \in Q | u(x, t) > k\}$. Then for any $h > k$, we get, from (2.8), that

$$(h - k)^2 \varphi(h)^{\frac{N}{N+2}} \leq \|(u - k)_+\|_{\frac{2(N+2)}{N}}^2 \leq e^{C\varrho_0} \varphi(k) k^2,$$

which then gives

$$\varphi(h) \leq e^{C\varrho_0} \left(\frac{k}{h - k} \right)^{\frac{2(N+2)}{N}} \varphi(k)^{\frac{N+2}{N}}. \quad (2.9)$$

Next, set $Y_s = \varphi(k_s)$, $k_s = M(2 - \frac{1}{2^s})$, and put $h = k_{s+1}$ and $k = k_s$ in (2.9) to get

$$Y_{s+1} \leq \tilde{c} 4^\tau (2^\tau)^s Y_s^{1+\varepsilon},$$

where $\tau = 2(N+2)/N$, $\varepsilon = 2/N$, and $\tilde{c} = e^{C\varrho_0}$. By Lemma 2.1, we get $\varphi(2M) = 0$ provided that

$$Y_0 = \varphi(k_0) = \varphi(M) \leq (\tilde{c} 4^\tau)^{-\frac{1}{\varepsilon}} (2^\tau)^{-\frac{1}{\varepsilon^2}}, \quad (2.10)$$

for some positive real number M . To determine the value of M , let $m > 1$ be an integer and $M = mK_0$, where K_0 is given by (2.5). Put $h = M = mK_0$ and $k = K_0$ in (2.9) to get

$$\varphi(M) \leq \tilde{c} \left(\frac{1}{m-1} \right)^\tau \varphi(K_0)^{1+\varepsilon} \leq \tilde{c} \left(\frac{1}{m-1} \right)^\tau T^{1+\varepsilon} (\text{meas} \Omega)^{1+\varepsilon}. \quad (2.11)$$

Now, to get $\varphi(2M) = 0$, we need to choose a proper m such that (2.10) holds. Combining (2.10) and (2.11), we only need the integer m to be such that

$$m \geq 1 + \tilde{c}^{\frac{1+\varepsilon}{\varepsilon\tau}} T^{\frac{1+\varepsilon}{\tau}} (\text{meas} \Omega)^{\frac{1+\varepsilon}{\tau}} 2^{\frac{2}{\varepsilon} + \frac{1}{\varepsilon^2}}.$$

Hence $\varphi(2M) = \varphi(2mK_0) = 0$ gives

$$u \leq 2mK_0 \leq e^{C\varrho_0} (\|F\|_\infty + |u_0|_\infty).$$

In a similar argument, we can also get the other half part of (2.3) for $-u$. This completes the proof. \square

3 Null controllability of a linear parabolic equation

In this section, we consider the null controllability of the linear parabolic equation

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (Bu) + \mathbf{1}_\omega f & \text{in } Q, \\ \partial_\nu u = 0 & \text{on } \Sigma, \\ u(x, 0) = u_0(x) & x \in \Omega. \end{cases} \quad (3.1)$$

Theorem 3.1. *Let $T > 0$, and $B \in L^\infty(Q)^N$ with $B \cdot \nu = 0$ on Σ . For any $u_0 \in L^2(\Omega)$, there exists a control $f \in L^\infty(Q_\omega)$ such that the solution u of system (3.1) corresponding to f satisfies $u \in W(0, T)$ and $u(x, T) = 0$ for $x \in \Omega$ almost everywhere. Moreover, the control f satisfies*

$$\|\mathbf{1}_\omega f\|_\infty \leq e^{C\kappa} |u_0|_2, \quad (3.2)$$

where

$$\kappa = (1 + \|B\|_\infty^2)(1 + T) + \frac{1}{T}. \quad (3.3)$$

To prove Theorem 3.1, we need to establish a type of “observability inequality” for the following adjoint equation of (3.1):

$$\begin{cases} -\partial_t \phi = \Delta \phi + B \cdot \nabla \phi & \text{in } Q, \\ \partial_\nu \phi = 0 & \text{on } \Sigma, \\ \phi(x, T) = \phi^T(x) & x \in \Omega, \end{cases} \quad (3.4)$$

where $\phi^T \in L^2(\Omega)$. By [8, Lemma 1.1], there is a function $\beta \in C^2(\overline{\Omega})$ such that $\beta(x) > 0$ for all $x \in \Omega$ and $\beta|_{\partial\Omega} = 0, |\nabla\beta(x)| > 0$ for all $x \in \overline{\Omega} \setminus \omega$. For $\lambda > 0$, set

$$\varphi = \frac{e^{\lambda\beta}}{t(T-t)}, \quad \alpha = \frac{e^{\lambda\beta} - e^{2\lambda\|\beta\|_{C(\overline{\Omega})}}}{t(T-t)}. \quad (3.5)$$

We then have a Carleman inequality stated in Lemma 3.1 (see [8]).

Lemma 3.1. *There exists a constant $\lambda_0 = \lambda_0(\Omega, \omega) > 1$ such that for all $\lambda \geq \lambda_0$ and $s \geq \gamma(\lambda)(T + T^2)$,*

$$\begin{aligned} & \iint_Q [s\varphi|\nabla y|^2 + (s\varphi)^3|y|^2] e^{2s\alpha} dxdt \\ & \leq C \iint_Q e^{2s\alpha} |\partial_t y \pm \Delta y|^2 dxdt + \iint_{Q_\omega} (s\varphi)^3 e^{2s\alpha} |y|^2 dxdt \end{aligned} \quad (3.6)$$

for all $y \in X = \{\xi \in C^{2,1}(\overline{Q}) | \partial_\nu \xi = 0 \text{ on } \Sigma\}$, where $\gamma(\lambda)$ is given by

$$\gamma(\lambda) = e^{2\lambda\|\beta\|_{C(\overline{\Omega})}}. \quad (3.7)$$

Proposition 3.1 is an “observability inequality” for the adjoint equation (3.4).

Proposition 3.1. *Let $\delta_0 \in (1, 2)$. Then there exist positive constants λ and s such that for all $T > 0, \phi^T \in L^2(\Omega)$, the solution ϕ of system (3.4) satisfies*

$$|\phi(\cdot, 0)|_2^2 \leq e^{C\kappa} \iint_{Q_\omega} e^{\delta_0 s\alpha} |\phi|^2 dxdt, \quad (3.8)$$

where κ is given by (3.3).

Proof. First, by Lemma 3.1, there exists a positive constant $\lambda_1 = C_1(\Omega, \omega)(1 + \|B\|_\infty^2)$ satisfying $\gamma(\lambda_1) \geq \lambda_1 > 1$ such that for any $\lambda \geq \lambda_1, s \geq \gamma(\lambda)(T + T^2)$ and $\phi^T \in L^2(\Omega)$, the associated solution ϕ to (3.4) satisfies

$$\iint_Q (s\varphi)^3 |\phi|^2 e^{2s\alpha} dxdt \leq C \iint_{Q_\omega} (s\varphi)^3 |\phi|^2 e^{2s\alpha} dxdt, \quad (3.9)$$

where $C = C(\Omega, \omega)$, and $\gamma(\lambda_1)$ and $\gamma(\lambda)$ are given by (3.7). By (3.4),

$$\frac{d}{dt} \left(e^{\|B\|_\infty^2 t} |\phi|_2^2 \right) \geq 0.$$

This gives, for any $t \in (0, T]$, that

$$|\phi(\cdot, 0)|_2^2 \leq e^{\|B\|_\infty^2 T} |\phi(\cdot, t)|_2^2 \quad (3.10)$$

for all $t \in (0, T]$. Integrate both sides of (3.10) over $[T/4, 3T/4]$ to give

$$|\phi(\cdot, 0)|_2^2 \leq \frac{2}{T} e^{\|B\|_\infty^2 T} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\Omega |\phi|^2 dxdt. \quad (3.11)$$

Since $(s\varphi)^{-3} e^{-2s\alpha} \leq e^{Cs/T^2}$ in $\Omega \times [T/4, 3T/4]$, inequality (3.8) then follows from (3.9) and (3.11), with λ and s taken as $\lambda = C(1 + \|B\|_\infty^2) > \lambda_1$ and $s = C(1 + \|B\|_\infty^2)(T + T^2)$. \square

Proof of Theorem 3.1. Let $\varepsilon > 0$. We set

$$J_\varepsilon(u, f) = \frac{1}{2} \iint_{Q_\omega} |f|^2 e^{-\delta_0 s \alpha} dx dt + \frac{1}{2\varepsilon} \int_\Omega |u(x, T)|^2 dx$$

and consider the extremal problem

$$\inf_{(u, f) \in \mathcal{U}} J_\varepsilon(u, f),$$

where \mathcal{U} is the totality of $(u, f) \in W(0, T) \times L^2(Q)$ solving (3.1). The existence of an optimal pair $(f_\varepsilon, u_\varepsilon)$ to the above extremal problem follows from a standard argument. By the maximum principle (see [4]), we get the optimality system for this problem as follows:

$$\begin{cases} -\partial_t \phi_\varepsilon = \Delta \phi_\varepsilon + B \cdot \nabla \phi_\varepsilon & \text{in } Q, \\ \partial_\nu \phi_\varepsilon = 0 & \text{on } \Sigma, \\ \phi_\varepsilon(x, T) = -\frac{1}{\varepsilon} u_\varepsilon(x, T) & x \in \Omega; \end{cases} \quad (3.12)$$

$$\begin{cases} \partial_t u_\varepsilon = \Delta u_\varepsilon - \nabla \cdot (B u_\varepsilon) + \mathbf{1}_\omega f_\varepsilon & \text{in } Q, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \Sigma, \\ u_\varepsilon(x, 0) = u_0(x) & x \in \Omega; \end{cases} \quad (3.13)$$

$$f_\varepsilon - \mathbf{1}_\omega \phi_\varepsilon e^{\delta_0 s \alpha} = 0. \quad (3.14)$$

Note that we can choose s and λ such that the observability inequality (3.8) and

$$\omega(\lambda) = e^{-\lambda \|\beta\|_{C(\overline{\Omega})}} < \delta_0 - 1 \quad (3.15)$$

hold. Furthermore, by (3.12), (3.13), (3.14), and Proposition 3.1, we get

$$\iint_{Q_\omega} |\phi_\varepsilon|^2 e^{\delta_0 s \alpha} dx dt + \frac{1}{\varepsilon} \int_\Omega |u_\varepsilon(x, T)|^2 dx \leq e^{C\kappa} |u_0|_2^2. \quad (3.16)$$

Eq. (3.14) and (3.16) then lead to $\|\mathbf{1}_\omega f_\varepsilon\|_2 \leq e^{C\kappa} |u_0|_2$, which means that the controls f_ε can be taken in L^2 space.

Now, we show that f_ε can actually be taken in L^∞ space. To this purpose, we apply a so-called bootstrap method in [17] (see also [5]). Firstly, set $\alpha_0 = \min_{\overline{\Omega}} \alpha$. Then the following inequalities can be easily verified:

$$\alpha_0 \leq \alpha \leq \frac{\alpha_0}{1 + \omega(\lambda)} < 0, \quad (3.17)$$

where $\omega(\lambda)$ is defined by (3.15). Secondly, let τ be a sufficiently small positive constant and let $\{\tau_j\}_{j=0}^M$ be a finite increasing sequence such that $0 < \tau_j < \tau, j = 0, 1, \dots, M, \tau_M = \tau$. Let $\{p_i\}_{i=0}^M$ be another finite increasing sequence such that $p_0 = 2, p_M = \infty$, and

$$-\left(\frac{N}{2} + 1\right) \left(\frac{1}{p_i} - \frac{1}{p_{i+1}}\right) + 1 > \frac{1}{2}, \quad i = 0, 1, \dots, M-1. \quad (3.18)$$

For each $i, i = 0, 1, \dots, M$, define

$$\begin{aligned} z_i(x, t) &= e^{(s+\tau_i)\alpha_0} \phi_\varepsilon(x, T-t), \\ F_i(x, t) &= \left[\partial_t (e^{(s+\tau_i)\alpha_0}) \right] \phi_\varepsilon(x, T-t), \\ \tilde{B}(x, t) &= B(x, T-t). \end{aligned}$$

Then, the adjoint equation (3.12) is transformed into the following initial-boundary problem for every $i = 0, 1, \dots, M$:

$$\begin{cases} \partial_t z_i - \Delta z_i = \tilde{B} \cdot \nabla z_i + F_i & \text{in } Q, \\ \partial_\nu z_i = 0 & \text{on } \Sigma, \\ z_i(x, 0) = 0 & x \in \Omega. \end{cases} \quad (3.19)$$

Let $\{S(t)\}_{t \geq 0}$ be the semigroup generated by the Laplace operator with Neumann boundary condition. It follows that (see, e.g., [3])

$$|S(t)u|_q \leq C m(t)^{-\frac{N}{2}(\frac{1}{p} - \frac{1}{q})} |u|_p \quad (3.20)$$

for all $u \in L^p(\Omega)$, $t > 0$, and $1 < p \leq q \leq \infty$, where $m(t) = \min\{1, t\}$. Note that the solution z_i of (3.19) can be represented as

$$z_i(\cdot, t) = \int_0^t S(t-s) \left(\tilde{B} \cdot \nabla z_i + F_i \right) (\cdot, s) ds, \quad i = 0, 1, \dots, M. \quad (3.21)$$

Applying the estimates (3.20) to (3.21), we have

$$|z_i(\cdot, t)|_{p_i} \leq C \int_0^t m(t-s)^{-\frac{N}{2}(\frac{1}{p_{i-1}} - \frac{1}{p_i})} \left| \left(\tilde{B} \cdot \nabla z_i + F_i \right) (\cdot, s) \right|_{p_{i-1}} ds, \quad i = 0, 1, \dots, M. \quad (3.22)$$

With (3.18), we apply Young's convolution inequality (see, e.g. [3]) to the right-hand side of (3.22) to get

$$\|z_i\|_{p_i} \leq e^{C(1+T)} \left(\|B\|_\infty \|\nabla z_i\|_{p_{i-1}} + \|F_i\|_{p_{i-1}} \right). \quad (3.23)$$

On the other hand, by a standard energy estimate applied to (3.19), we can get the following $L^{p_{i-1}}$ -estimate for z_i :

$$\|z_i\|_{p_{i-1}} + \|\nabla z_i\|_{p_{i-1}} \leq e^{C\kappa} \|F_i\|_{p_{i-1}}. \quad (3.24)$$

By the definition of F_i and (3.17), we have

$$\|F_i\|_{p_{i-1}} \leq CT \|z_{i-1}\|_{p_{i-1}}. \quad (3.25)$$

Combining (3.23), (3.24), and (3.25) then, we obtain

$$\|z_i\|_{p_i} \leq e^{C\kappa} \|z_{i-1}\|_{p_{i-1}}.$$

This iteration inequality from 0 to M produces

$$\|z_M\|_{p_M} \leq e^{C\kappa} \|z_0\|_2. \quad (3.26)$$

Since $p_M = \infty$, it follow from the definition of z_0 , (3.16), and (3.26) that $\|z_M\|_\infty \leq e^{C\kappa} \|u_0\|_2$; that is, $\|\phi_\varepsilon e^{(s+\tau)\alpha_0}\|_\infty \leq e^{C\kappa} \|u_0\|_2$. By (3.14), we get

$$\left\| e^{[-s(\delta_0 - 1 - \omega(\lambda)) + \tau(1 + \omega(\lambda))]\alpha} \mathbf{1}_\omega f_\varepsilon \right\|_\infty \leq e^{C\kappa} \|u_0\|_2, \quad (3.27)$$

where $\omega(\lambda)$ is given by (3.15). By choosing τ small enough so that $-s(\delta_0 - 1 - \omega(\lambda)) + \tau(1 + \omega(\lambda)) < 0$, we conclude from (3.27) that

$$\|\mathbf{1}_\omega f_\varepsilon\|_\infty \leq e^{C\kappa} \|u_0\|_2. \quad (3.28)$$

This shows that the controls f_ε can be taken in L^∞ space.

Finally, by (3.28), we can extract a subsequences of $\{f_\varepsilon\}_{\varepsilon \geq 0}$, still denoted by itself, such that $\mathbf{1}_\omega f_\varepsilon \rightarrow \mathbf{1}_\omega f \in L^\infty(Q)$ weakly in $L^2(Q)$ as $\varepsilon \rightarrow 0$. Denote by u_ε the solution to the system (3.13) associated to f_ε . By virtue of Proposition 2.2, $\{u_\varepsilon\}_{\varepsilon \geq 0}$ is uniformly bounded in $W(0, T)$. Thus, we can extract a subsequence of $\{u_\varepsilon\}_{\varepsilon \geq 0}$, still denoted by itself, such that $u_\varepsilon \rightarrow u$ weakly in $W(0, T)$ for $u \in W(0, T) \subset C([0, T]; L^2(\Omega))$. Such a u is the weak solution of (3.1) corresponding to f . In addition, by (3.16), $u(x, T) = 0$ almost everywhere in Ω . This completes the proof. \square

4 Proof of Theorem 1.1

Let $K = \{\xi \in L^\infty(Q) \mid \|\xi\|_\infty \leq 1\} \cap L^\infty(0, T; L^p(\Omega)) \subset L^2(Q)$, $p > \max\{N, 2\}$. For every $\xi \in K$, consider the following two linear equations:

$$\begin{cases} 0 = \Delta v(\cdot, t) - \gamma v(\cdot, t) + \delta \xi(\cdot, t) & \text{in } \Omega, \\ \partial_\nu v(\cdot, t) = 0 & \text{on } \partial\Omega \end{cases} \quad (4.1)$$

for almost every $t \in [0, T]$ and

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (Bu) + \mathbf{1}_\omega f & \text{in } Q, \\ \partial_\nu u = 0 & \text{on } \Sigma, \\ u(x, 0) = u_0(x) & x \in \Omega, \end{cases} \quad (4.2)$$

where $B = \chi \nabla v^\xi$. In what follows, we denote by shorthand $v^\xi = v^\xi(\cdot, t)$ the unique solution of equation (4.1) corresponding to $\xi(\cdot, t)$ for $t \in [0, T]$. First, by Proposition 2.1, we see that $v^\xi \in L^\infty(0, T; W^{2,p}(\Omega))$ for $p > \max\{N, 2\}$ provided that $\xi \in K$. Hence, the embedding theory between Sobolev spaces for $p > N$ (see, e.g., [6]) then implies

$$B = \chi \nabla v^\xi \in L^\infty(0, T; W^{1,p}(\Omega))^N \subset L^\infty(0, T; C(\bar{\Omega}))^N \text{ with } B \cdot \nu = 0 \text{ on } \Sigma,$$

and in addition,

$$\|B\|_\infty = \chi \left\| \nabla v^\xi \right\|_{L^\infty(0, T; W^{1,p}(\Omega))} \leq C \|\xi\|_\infty \leq C. \quad (4.3)$$

Thus, we can define a linear continuous operator Φ from K to $L^\infty(0, T; C(\bar{\Omega}))^N \subset L^\infty(Q)^N$ by

$$\Phi(\xi) = B = \chi \nabla v^\xi, \forall \xi \in K.$$

Second, by Theorem 3.1, we see that for each $B \in L^\infty(Q)^N$ with $B \cdot \nu = 0$ on Σ , there exists a pair $(u, f) \in L^2(Q) \times L^\infty(Q)$ that solves system (4.2) with $u(x, T) = 0$ for almost all $x \in \Omega$. Moreover, the control f satisfies (3.2). By (4.3),

$$\|\mathbf{1}_\omega f\|_\infty \leq e^{C\kappa_0} |u_0|_2, \quad (4.4)$$

where

$$\kappa_0 = 1 + T + \frac{1}{T}. \quad (4.5)$$

and in the sequel, C is a positive constant independent of time T . By (2.5) in Proposition 2.2 and (4.4), we have the following estimate:

$$\|u\|_{W(0,T)} + \|u\|_{\infty} \leq e^{C\kappa_0} |u_0|_{\infty}. \quad (4.6)$$

We then define a multi-valued mapping $\Psi : L^{\infty}(Q)^N \rightarrow 2^{L^2(Q)}$ by

$$\Psi(B) = \left\{ u \in L^2(Q) \mid \begin{array}{l} \exists f \in L^{\infty}(Q_{\omega}) \text{ satisfying (4.4) such that } u \text{ is the solution} \\ \text{of (4.2) corresponding to } f \text{ and } B, \text{ and } u(x, T) = 0 \text{ a.e. in } \Omega \end{array} \right\}$$

where $2^{L^2(Q)}$ stands for all subsets of $L^2(Q)$. Since both operators Φ and Ψ are well defined, which is guaranteed by Proposition 2.1 and Theorem 3.1, we let

$$\Lambda = \Psi \circ \Phi : K \subset L^2(Q) \rightarrow 2^{L^2(Q)}. \quad (4.7)$$

Now, we apply Kakutani's fixed point theorem (see [4, p.7]) to the map Λ to prove Theorem 1.1. Indeed, it is clear that K is a convex subset of $L^2(Q)$. By Proposition 2.1 and Theorem 3.1 again, for any $\xi \in K$, $\Lambda(\xi)$ is nonempty and it is also convex due to the linearity of the equations. Moreover, from (4.6), it follows that for each $\xi \in K$, $\Lambda(\xi)$ is bounded in $W(0, T)$, and hence a compact subset of $L^2(Q)$ according to the Aubin-Lions lemma (see [4, p.17]).

We claim that Λ is upper semi-continuous. Indeed, let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of functions in K such that

$$\xi_n \rightarrow \xi \text{ strongly in } L^2(Q) \text{ as } n \rightarrow \infty. \quad (4.8)$$

For every n , let $B_n = \Phi(\xi_n) = \chi \nabla v_n$ and take $u_n \in \Lambda(\xi_n) = \Psi(B_n)$, where v_n solves

$$\begin{cases} 0 = \Delta v_n(\cdot, t) - \gamma v_n(\cdot, t) + \delta \xi_n(\cdot, t) & \text{in } \Omega, \\ \partial_{\nu} v_n(\cdot, t) = 0 & \text{on } \partial\Omega \end{cases} \quad (4.9)$$

for almost every $t \in [0, T]$ and u_n solves

$$\begin{cases} \partial_t u_n = \Delta u_n - \nabla \cdot (B_n u_n) + \mathbf{1}_{\omega} f_n & \text{in } Q, \\ \partial_{\nu} u_n = 0 & \text{on } \Sigma, \\ u_n(x, 0) = u_0(x) & x \in \Omega, \end{cases} \quad (4.10)$$

with $u_n(x, T) = 0$ for almost all $x \in \Omega$. Moreover, the control f_n satisfies

$$\|\mathbf{1}_{\omega} f_n\|_{\infty} \leq e^{C\kappa_0} |u_0|_2. \quad (4.11)$$

To show that Λ is upper semi-continuous, it suffices to prove that there exist a subsequence of $\{u_n\}_{n=1}^{\infty}$ such that it converges strongly to an element of $\Lambda(\eta)$ in $L^2(Q)$ topology.

In what follows, we do not distinguish the sequence and its subsequence by abuse of notation. First, the estimate (4.11) enables us to obtain a function $f \in L^{\infty}(Q)$ and a subsequence of $\{f_n\}_{n=1}^{\infty}$, such that

$$\mathbf{1}_{\omega} f_n \rightarrow \mathbf{1}_{\omega} f \text{ weakly in } L^2(Q); \text{ weakly}^* \text{ in } L^{\infty}(Q) \text{ as } n \rightarrow \infty. \quad (4.12)$$

By (4.11) and Proposition 2.2, u_n satisfies (4.6); that is

$$\|u_n\|_{W(0,T)} + \|u_n\|_{\infty} \leq e^{C\kappa_0} |u_0|_2. \quad (4.13)$$

Applying the Aubin-Lions lemma again, we get a $u \in W(0,T) \cap L^{\infty}(Q)$ and a subsequence of $\{u_n\}_{n=1}^{\infty}$ such that

$$u_n \rightarrow u \text{ weakly in } W(0,T); \text{ strongly in } L^2(Q), \text{ as } n \rightarrow \infty. \quad (4.14)$$

Furthermore, by the strong convergence of $\{u_n\}_{n=1}^{\infty}$ in $L^2(Q)$, we can extract a subsequence of $\{u_n\}_{n=1}^{\infty}$ (see [15, Lemma 2.1, p. 72]) such that

$$u_n \rightarrow u \text{ almost everywhere in } Q \text{ as } n \rightarrow \infty. \quad (4.15)$$

On the other hand, by Proposition 2.1, for each n and $p > 1$, it holds that

$$\|v_n(\cdot, t)\|_{W^{2,p}(\Omega)} \leq C \|\xi_n(\cdot, t)\|_{L^p(\Omega)} \leq C, \text{ for almost every } t \in [0, T] \quad (4.16)$$

Thus, B_n satisfies (4.3); that is

$$\|B_n\|_{\infty} \leq C \|\nabla v_n\|_{L^{\infty}(0,T;W^{1,p}(\Omega))} \leq C \|\xi_n\|_{\infty} \leq C. \quad (4.17)$$

Furthermore, let v be the unique solution of (4.1) corresponding to ξ . Then, by the linearity of Eq. (4.1) and by (4.16),

$$\|v_n(\cdot, t) - v(\cdot, t)\|_{H^2(\Omega)} \leq C \|\xi_n(\cdot, t) - \xi(\cdot, t)\|_2, \text{ for almost every } t \in [0, T].$$

Since $\xi_n \rightarrow \eta$ strongly in $L^2(Q)$ in condition (4.8), it follows that

$$\|v_n - v\|_{L^2(0,T;H^2(\Omega))} \leq C \|\xi_n - \xi\|_2 \rightarrow 0,$$

which implies that

$$B_n = \chi \nabla v_n \rightarrow \chi \nabla v = B \text{ strongly in } L^2(Q). \quad (4.18)$$

By (4.14) and (4.17), the sequence $\{B_n u_n\}_{n=1}^{\infty}$ is bounded in $L^2(Q)$, whence there is a subsequence such that $B_n u_n \rightarrow \pi$ weakly in $L^2(Q)$ as $n \rightarrow \infty$. By (4.18), there is a subsequence of $\{B_n\}_{n=1}^{\infty}$ such that $B_n \rightarrow B$ almost everywhere in Q . This together with (4.15) implies that $B_n u_n \rightarrow Bu$ almost everywhere in Q . Therefore $\pi = Bu$, and

$$B_n u_n \rightarrow Bu \text{ weakly in } L^2(Q) \text{ as } n \rightarrow \infty. \quad (4.19)$$

Now, by (4.12), (4.14), and (4.19), we can pass to the limit as $n \rightarrow \infty$ in (4.10) to get that u is a weak solution of (4.2) corresponding to B and f in the sense of Definition 1. Now, it only need to show that $u \in \Lambda(\xi) = \Psi(B)$. Actually, let $U_n = u_n - u$ and $F_n = \mathbf{1}_{\omega}(f_n - f)$. Then U_n solves the system

$$\begin{cases} \partial_t U_n = \Delta U_n - \nabla \cdot (B_n U_n) - \nabla \cdot [(B_n - B)u] + F_n & \text{in } Q, \\ \partial_{\nu} U_n = 0 & \text{on } \Sigma, \\ U_n(x, 0) = 0 & x \in \Omega. \end{cases} \quad (4.20)$$

Multiply the first equation of (4.20) by U_n , and integrate over Ω to give

$$\frac{d}{dt} |U_n|_2^2 + |\nabla U_n|_2^2 \leq C \|B_n\|_\infty^2 |U_n|_2^2 + C \|u\|_\infty^2 |B_n - B|_2^2 + C \int_\Omega F_n Y_n dx.$$

It then follows from Gronwall's lemma that

$$|U_n(\cdot, T)|_2^2 \leq e^{C\|B_n\|_\infty^2 T} \left(C \|u\|_\infty^2 \|B_n - B\|_2^2 + C \int_\Omega F_n Y_n dx \right). \quad (4.21)$$

By (4.12), (4.17), and (4.18), we see that the right-hand side of (4.21) tends to 0 as $n \rightarrow \infty$, and hence $|U_n(\cdot, T)|_2 \rightarrow 0$. Since $u_n(x, T) = 0$ for almost all $x \in \Omega$, we get that $u(x, T) = 0$ for almost all $x \in \Omega$. It then follows that $u \in \Psi(B) = \Lambda(\xi)$. Therefore, Λ is upper semi-continuous.

Finally, it remains to show that $\Lambda(K) \subset K$. Indeed, by the standard energy estimate we see that for any $\xi \in K$, each element u of $\Lambda(\xi)$ satisfies

$$\|u\|_{L^\infty(0, T; L^p(\Omega))} \leq e^{C(1+\|B\|_\infty^2)(1+T)} \left(|u_0|_p + \|\mathbf{1}_\omega f\|_p \right),$$

which together with (2.5) in Proposition 2.2 and (4.4) leads to $u \in L^\infty(Q) \cap L^\infty(0, T; L^p(\Omega))$ and

$$\|u\|_{L^\infty(0, T; L^p(\Omega))} + \|u\|_\infty \leq e^{c_1 \kappa_0} |u_0|_\infty,$$

where c_1 is a positive constant independent of T , and κ_0 is given by (4.5). If $|u_0|_\infty \leq e^{-c_1 \kappa_0}$ which is exactly (1.2), then $\|u\|_\infty \leq 1$ and hence $\Lambda(K) \subset K$. Apply Kakutani's fixed point theorem to obtain at least one fixed point u of Λ ; that is $u \in \Lambda(u)$. This u together with $v = v_u$, the solution of (4.1) with $\xi = u$, gives the solution of (1.1), corresponding with some control f and $u(x, T) \equiv 0$. This completes the proof. \square

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